Invariancy of Total Shear Stress for Compressible Turbulent Flows

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IN the present note, a proof of the invariancy of the tota shear stress in free turbulent shear flows is given. It is pointed out that some of the more stringent restrictions imposed by other investigators are not necessary. The Howarth[1] transformation is used to transform the compressible flow field to an incompressible field, and the invariancy of the stream function under this transformation is assumed.

The analysis of compressible turbulent free shear flows is a complex problem. One of the main difficulties in transforming the compressible turbulent flow equations to the incompressible form lies in the transformation of the shear stress term. Some time ago, Mager[2] showed that the Howarth transformation is valid for the compressible turbulent boundary-layer-type problems. However, in applying the transformation he made the assumption that the turbulent shear stresses are invariant under the transformation. Recently Burggraf[3] relaxed the more stringent assumption of Mager by requiring only the y derivative of the turbulent shear to be invariant to permit the use of the transformation. In the present note, the authors show local invariance of the total shear stress across sections perpendicular to the flow direction.

Transformations

The well-known turbulent boundary-layer equations for compressible flows with constant pressure are given by

$$(\partial/\partial x)\bar{\rho}\bar{u} + (\partial/\partial y)\overline{\rho v} = 0 \tag{1}$$

$$\bar{\rho}\bar{u}(\partial\bar{u}/\partial x) + \overline{\rho v}(\partial\bar{u}/\partial y) = (\partial/\partial y)[(\bar{\mu}\partial\bar{u}/\partial y) - \bar{\rho}(\overline{v'u'})] \tag{2}$$

Introducing Howarth's transformation, we get

$$x_* = x \qquad y_* = \int_0^y \frac{\bar{\rho}}{\rho_*}\, dy \tag{3}$$

where ρ_* is a reference density taken to be a constant for a given flow.

Now we define a stream function ψ, which is assumed to be invariant under the previous transformation ($\psi \equiv \psi_*$), as

$$\bar{\rho}\bar{u} = \rho_*(\partial\psi/\partial y) \qquad \overline{\rho v} = -\rho_*(\partial\psi/\partial x) \tag{4}$$

for incompressible flow, or in the transformed plane we have

$$u_* = \partial\psi_*/\partial y_* \qquad v_* = -(\partial\psi_*/\partial x_*) \tag{5}$$

Upon integration of the momentum equation (2), we have

$$\tau = \mu\frac{\partial\bar{u}}{\partial y} - \bar{\rho}(\overline{v'u'}) = \int_0^y \left[\bar{\rho}\bar{u}\frac{\partial\bar{u}}{\partial x} + \overline{\rho v}\frac{\partial\bar{u}}{\partial y}\right] dy + A_0(x) \tag{6}$$

where $A_0(x)$ is a constant of integration. Now applying the transformation on an arbitrary function F we have, from Ref. 3,

$$\bar{\rho}\bar{u}(\partial F/\partial x) + \overline{\rho v}(\partial F/\partial y) = \bar{\rho}[u_*(\partial F/\partial x_*) + v_*(\partial F/\partial y_*)] \tag{7}$$

Hence Eq. (6) in the transformed plane takes the form

$$\tau = \rho_* \int_0^{y_*}\left[u_*\frac{\partial u_*}{\partial x_*} + v_*\frac{\partial u_*}{\partial y_*}\right] dy_* + A_0(x) \tag{8}$$

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where the upper limit of integration is the value of y_* corresponding to the upper limit in Eq. (6).

Now integration of the incompressible momentum equation between the same limits of integration as Eq. (8) gives the result

$$\tau_* = \rho_* \int_0^{y_*}\left[u_*\frac{\partial u_*}{\partial x_*} + v_*\frac{\partial u_*}{\partial y_*}\right] dy_* + A_0{}^*(x_*) \tag{9}$$

Subtracting Eqs. (8) and (9) gives the result

$$\tau - \tau_* = A_0(x) - A_0{}^*(x_*) = A(x) \tag{10}$$

The total shear stress is thus locally invariant across any axial (x = const) section under the compressibility transformation.

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Effect of Pulse Shape on Final Deformation of Spherical Shells under Impulsive Loading

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Introduction

WHEN an ideal rigid plastic structure is subjected to a load greater than the static collapse pressure, no equilibrium configuration of the stresses can exist, and the structure will accelerate. Obviously, if such a load is applied for any appreciable time, the resulting deformations will become very large, and the structure will no longer be serviceable. However, if the load is applied for an extremely short time, the inertial resistance of the structure may be sufficient to prevent excessive motion, and the usefulness of the structure may not be impaired.

In an earlier investigation[1] the behavior of a spherical shell under stepwise impulsive loading was considered. The present note is concerned with studying the effect of pulse shape on the final deformation of plastic spherical shells. The material of the shell is assumed to be rigid, perfectly plastic, and to obey Tresca's yield condition and the associated flow rule. The load is assumed to be greater than the static collapse pressure and to act for a short period of time. Specifically, triangular and square waves are considered. It is found that the pulse shape has a profound effect on the final deformation of the shell, even though the peak load and the total impulse of the two waves are adjusted to have the same value. The final deformations of the shell are shown graphically as a function of the peak load.

Basic Equations

The state of stress in a rotationally symmetric shell is described by four generalized stresses: the circumferential and meridional bending moments M_θ and M_ϕ and the circumferential and meridional membrane forces N_θ and N_ϕ.

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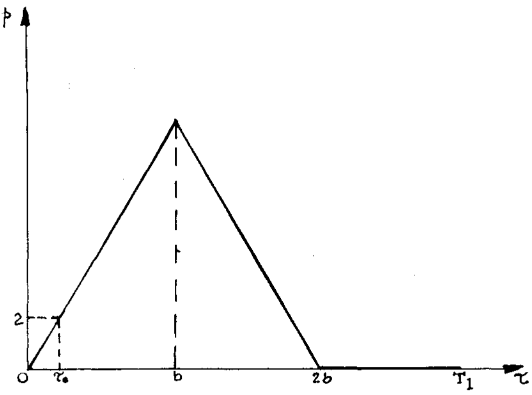

Fig. 1 Triangular wave.

Similarly, the state of strain is described by four generalized strains, which may, in turn, be expressed in terms of the meridional and normal components of the displacements V and W.

The plastic yield condition for a rotationally symmetric shell based on the Tresca criterion has been described by Onat and Prager.[2] The yield surface is a closed convex hypersurface in the four-dimensional stress space, with the rectangular Cartesian coordinates M_θ, M_ϕ, N_θ, and N_ϕ.

If the load is applied impulsively, the generalized stresses must satisfy three equations of motion which may be written

$$\left.\begin{aligned} &(n_\phi \sin\phi)' - n_\theta \cos\phi = \sin\phi(s + \gamma\ddot{v}) \\ &(s \sin\phi)' + \sin\phi(p + n_\theta + n_\phi)' = \gamma\ddot{w} \sin\phi \\ &k[(m_\phi \sin\phi)' - m_\theta \cos\phi] = s \sin\phi \end{aligned}\right\} \quad (1)$$

where we have defined

$$n = N/N_0 = N/2\sigma_0 H \qquad m = M/M_0 = M/\sigma_0 H^2$$
$$s = S/N_0 \qquad p = RP/N_0 \qquad k = M_0/RN_0$$
$$v = V/R \qquad w = W/R$$
$$\tau = t/T_0 \qquad \gamma = \rho R^2/N_0 T_0^2$$

σ_0 is the tensile yield stress of the material, and ρ is the surface density. The shell is of uniform thickness $2H$ and radius R and is subjected to a uniform radial pressure P. Primes and dots denote differentiation with respect to ϕ and τ, respectively. The generalized strain rates and the velocities are related by

$$\left.\begin{aligned} &\epsilon_\theta = \dot{v} \cot\phi - \dot{w} \qquad \epsilon_\phi = \dot{v}' - \dot{w} \\ &\varkappa_\theta = -k \cot\phi(\dot{v} + \dot{w}') \qquad \varkappa_\phi = -k(\dot{v} + \dot{w}')' \end{aligned}\right\} \quad (2)$$

Onat and Prager[2] have defined the yield surface for a rotationally symmetric shell in terms of the parameters p, q, and r, which are related to the strain rates by

$$p = -\epsilon_\theta/h\varkappa_\theta \qquad q = -(\epsilon_\theta + \epsilon_\phi)/h(\varkappa_\theta + \varkappa_\phi)$$
$$r = -\epsilon_\phi/h\varkappa_\phi$$

The yield surface consists of several regions, which are distinguished by the relative magnitudes of p, q, and r.

Finally, the initial and continuity conditions must be stated. Since the shell is assumed to be at rest until p reaches the value of p_{static}, the initial conditions at $\tau_0 = p_{\text{static}/c}$ are

$$w(\phi, \tau_0) = \dot{w}(\phi, \tau_0) = 0 \quad (3)$$

w and $\dot{w}$ must be continuous throughout. In Eq. (3), c represents the slope of the pressure-time curve.

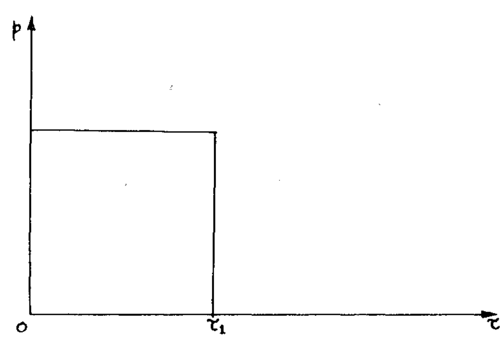

Fig. 2 Square wave.

Solution

If the magnitude of the pressure is less than the static collapse pressure, then no motion will take place. At the instant of collapse, the solution is found to be[1]

$$n_\theta = n_\phi = -1 \qquad m_\theta = m_\phi = 0 \quad (4a)$$
$$\dot{w} = \dot{w}_0 \qquad \dot{v} = 0 \quad (4b)$$
$$p_{\text{static}} = 2 \quad (4c)$$

where $\dot{w}_0$ is the constant radial velocity. The solution given by Eqs. (4) is statically and kinematically admissible provided that $\dot{w}_0 > 0$.

The incipient plastic flow that takes place under p_{static} is quasi-static, and inertia forces do not arise. If the applied load is greater than the static collapse pressure, accelerated motion of the shell takes place. For values of the load which are only slightly greater than the static collapse pressure, it is reasonable to expect that the same plastic regime will hold good as that in the static case. This hypothesis is tested by solving Eqs. (1) and (4a). The resulting radial acceleration is given by

$$\gamma\ddot{w} = p - 2 \quad (5)$$

For the triangular wave, Fig. 1, the pressure-time relationship is given by

$$\left.\begin{aligned} &p = c\tau && 0 < \tau < b \\ &p = c(2b - \tau) && b < \tau < 2b \\ &p = 0 && \tau > 2b \end{aligned}\right\} \quad (6)$$

Substituting Eqs. (6) into Eq. (5) and integrating subject to the initial and continuity conditions, Eqs. (3) yield

$$\left.\begin{aligned} &\gamma\ddot{w} = c\tau - 2 \\ &\gamma\dot{w} = (c\tau - 2)^2/2c \\ &\gamma w = (c\tau - 2)^3/6c^2 \end{aligned}\right\} \tau_0 < \tau < b \quad (7a)$$

$$\left.\begin{aligned} &\gamma\ddot{w} = c(2b - \tau) - 2 \\ &\gamma\dot{w} = (1/2c)[-c^2(2b - \tau)^2 - 4c\tau + 2(b^2c^2 + 2)] \\ &\gamma w = (1/6c^2)[c^3(2b - \tau)^3 - 6c^2\tau^2 + 6c\tau(b^2c^2 + 2) - 2(3b^3c^3 + 4)] \end{aligned}\right\} b < \tau < 2b \quad (7b)$$

$$\left.\begin{aligned} &\gamma\ddot{w} = -2 \\ &\gamma\dot{w} = (1/c)(-2c\tau + b^2c^2 + 2) \\ &\gamma w = (1/6c^2)[-6c^2\tau^2 + 6c\tau(b^2c^2 + 2) - 2(3b^3c^3 + 4)] \end{aligned}\right\} 2b < \tau < T_1 \quad (7c)$$

where

$$T_1 = (b^2c^2 + 2)/2c \quad (8a)$$

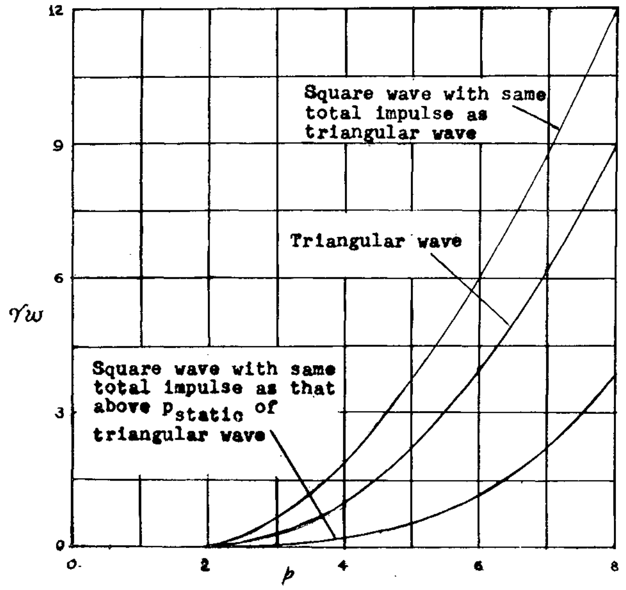

Fig. 3 Comparison of displacements resulting from triangular and square waves.

The condition that $T_1 > 2b$ puts a limitation on the values of p that can be applied, namely,

$$p > 2 + 2^{1/2} \tag{8b}$$

For the squares wave, Fig. 2, a pressure-time relationship is given by

$$p > p_{\text{static}} \qquad 0 < \tau < \tau_1 \tag{9a}$$

$$p = 0 \qquad \tau_1 < \tau < T_2 \tag{9b}$$

where

$$T_2 = p\tau_1/2 \tag{10}$$

Substituting Eqs. (9) into Eq. (5) and integrating subject to the initial and continuity conditions, Eqs. (3) yield

$$\left.\begin{aligned} \gamma\ddot{w} &= p - 2 \\ \gamma\dot{w} &= (p - 2)\tau \\ \gamma w &= (p - 2)(\tau^2/2) \end{aligned}\right\} 0 < \tau < \tau_1 \tag{11a}$$

$$\left.\begin{aligned} \gamma\ddot{w} &= -2 \\ \gamma\dot{w} &= (p\tau_1 - 2\tau) \\ \gamma w &= (p\tau_1/2)(2\tau - \tau_1) - \tau^2 \end{aligned}\right\} \tau_1 < \tau < T_2 \tag{11b}$$

It should be noted that for the square wave $c = \infty$, and hence $\tau_0 = 0$. The solution for the triangular wave is compared with that of two matching square waves, all with the same peak load. If the square wave has the same total impulse as the triangular wave, we get

$$\tau_1 = b \tag{12a}$$

If the square wave is chosen so that the triangular wave has the same impulse (due to that part of the pressure which exceeds p_{static}) as that of the square wave, we get

$$\tau_1 = (bc - 2)^2/bc^2 \tag{12b}$$

The final displacements due to the triangular wave and the two matching square waves are plotted in Fig. 3. It is seen that, when the square wave has the same total impulse as the triangular wave, the displacement is considerably overestimated. On the other hand, when the waves are chosen so that they have the same impulse due to that part of the pressure which exceeds p_{static}, the displacement is considerably less. Similar results have been obtained by Hodge[3] for the case of circular cylindrical shells.

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Validity of Triple-Point Calculation Applied to Jet Mach Disk

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Recently a general description was presented and published for a method of calculating the size and position of the Mach disk in an underexpanded inviscid jet.[1] The technique is applicable to calculating the mixed subsonic-supersonic flow downstream of the Mach disk. The authors presented the results of a calculation for one particular set of flow conditions.[1] They compared their result with a schlieren picture for the same set of experimental flow conditions. The agreement was good.

To utilize the technique[1] for calculating the position of the Mach disk one must first calculate the inviscid flow field without the presence of the Mach disk (for example, by the method of characteristics). The location of the Mach disk on the interior or jet shock is ascertained by applying the usual triple-point equations and by requiring that the Mach disk be locally normal to the incident flow at the triple point.

Adamson[2] has questioned the range of validity of this approach based upon the results of Kawamura.[3] Kawamura's results indicate that the triple-point calculation does not agree with experiment except for a small range of stagnation-to-ambient-pressure ratios near the value where the Mach disk first occurs. At the highest pressure ratio treated by Kawamura, he was not able to find a triple-point solution consistent with the experimental results. Adamson[2] suggested that the normal triple-point relations cannot be applied to this higher pressure ratio. As a result of this apparent discrepancy, this particular case has been studied in some detail.

The version of the Kawamura paper studied by the present authors did not contain schlieren photos suitable for analysis. The present authors duplicated the experimental apparatus of Kawamura. Results were obtained. The quality of the schlieren photos was not as good as that given by Ladenburg and Bershader[4] for a smaller orifice diameter but with nearly the same pressure ratio as the one that gave Kawamura trouble. The results obtained by the present authors and by Ladenburg and Bershader are similar. It is presumed that they agree also with the results obtained by Kawamura.

Kawamura found the triple-point relations invalid for a total-to-ambient-pressure ratio (p_t/p_s) equal to 5.72. In Fig. 1 is given the shadowgraph result of Ladenburg and Bershader for p_t/p_a equal to 5.72. Also shown in Fig. 1 are the results from the triple-point calculation made by the present authors assuming that the Mach disk is locally perpendicular

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